Art - Reciprocal property of Adjoint Equation

If $\overline{L_n}(y) = 0$ — ① is the Adjoint equation of $L_n(y) = 0$ — ②

then adjoint equation $\overline{L_n}(y) = 0$ is the equation $L_n(y) = 0$

This is called reciprocal property of Adjoint equation.

Art - Self - Adjoint

An nth order linear equation $L_n(y) = 0$ is said to be Self Adjoint if it is identical with its Adjoint equation.

i.e $\overline{L_n}(y) = L_n(y)$

Ex - check the equation for self Adjoint

$$y'' - xy = 0 \quad \text{—— ①}$$

$$\rightarrow \quad L_2(y) = \frac{d^2y}{dx^2} - xy = 0$$

$$\overline{L_2}(y) = (-1)^2 \frac{d^2}{dx^2}(y) + (-1)^1 \frac{d}{dx}(0 \cdot y) - xy = 0$$

$$\bar{L}_2(y) = \frac{d^2y}{dx^2} - xy = 0$$

$$L_2(y) = \bar{L}_2(y)$$

The given linear equation is self Adjoint

Ex - Show that $L_2(y) = (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$ is self Adjoint.

$$\Rightarrow \quad \bar{L}_2(y) = (-1)^2\frac{d^2}{dx^2}\left((1-x^2)y\right) + (-1)^1\frac{d}{dx}(-2xy)$$

$$+ n(n+1)y = 0$$

$$= \frac{d}{dx}\left[(1-x^2)\frac{dy}{dx} - 2xy\right] + \left[2x\frac{dy}{dx} + 2y\right] + n(n+1)y = 0$$

$$= (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 2x\frac{dy}{dx} - 2y + 2x\frac{dy}{dx} + 2y$$

$$+ n(n+1)y = 0$$

$$= (1-x^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 2x\frac{dy}{dx} + n(n+1)y = 0$$

$$= (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

Since $\bar{L}_2(y) = L_2(y)$

The given linear equation is self-Adjoint

Theorem : The necessary and sufficient condition for the equation

$$L_2(y) = a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \quad \text{--- (1)}$$

be a self-Adjoint that

$$a_0'(x) = a_1(x) \quad \text{--- (2)}$$

→ The adjoint of given linear eqn is

$$\bar{L}_2(y) = (-1)^2\frac{d^2}{dx^2}[a_0(x)y] + (-1)^1\frac{d}{dx}[a_1(x)y] + a_2(x)y = 0$$

$$= \frac{d}{dx}\left[a_0(x)\frac{dy}{dx} + a_0'(x)y\right] - \frac{d}{dx}[a_1(x)y] + a_2(x)y = 0$$

$$= a_0(x)\frac{d^2y}{dx^2} + a_0'(x)\frac{dy}{dx} + a_0'(x)\frac{dy}{dx} + a_0''(x)y - \left[a_1(x)\frac{dy}{dx} + a_1'(x)\right.$$

$$+ a_2(x)y = 0$$

$$= a_0(x)\frac{d^2y}{dx^2} + [2a_0'(x) - a_1(x)]\frac{dy}{dx} + [a_0''(x) - a_1'(x) + a_2(x)]y = 0 \quad \text{—— (3)}$$

<u>Condition is Necessary</u>

Given that equation (1) is Self Adjoint

$$\bar{L}_2(y) = L_2(y)$$

comparing (1) and (3)

$$2a_0'(x) - a_1(x) = a_1(x)$$

$$2a_0'(x) = 2a_1(x)$$

$$a_0'(x) = a_1(x) \quad \text{—— (4)}$$

$$a_0''(x) - a_1'(x) + a_2(x) = a_2(x)$$

$$a_0''(x) - a_1'(x) = 0$$

$$a_0''(x) = a_1'(x) \quad \text{—— (5)}$$

Integrating (5)

$$a_0'(x) = a_1(x) + C \quad \text{—— (6)}$$

Putting in eqⁿ (4)

$a_1(x) + c = a_1(x)$

$c = 0$

Put in (6)

$$\boxed{a_0'(x) = a_1(x)}$$

<u>condition</u> is <u>Sufficient</u>

Given that $a_0'(x) = a_1(x)$

Differentiate wrt x

$a_0''(x) = a_1'(x)$

Put in (3)

we get,

$$\bar{L}_2(y) = a_0(x)\frac{d^2y}{dx^2} + [2a_1(x) - a_1(x)]\frac{dy}{dx} + [a_1'(x) - a_1'(x) + a_2(x)]y = 0$$

$$= a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

$\bar{L}_2(y) = L_2(y)$

$L_2(y)$ is self Adjoint